CHAP. VIII.

Of the Properties *of* Fractions.

85. We have already seen, that each of the fractions,

$$\tfrac{2}{2}, \tfrac{3}{3}, \tfrac{4}{4}, \tfrac{5}{5}, \tfrac{6}{6}, \tfrac{7}{7}, \tfrac{8}{8}, \text{\&c.}$$

makes an integer, and that consequently they are all equal to one another. The same equality prevails in the following fractions,

$$\tfrac{2}{1}, \tfrac{4}{2}, \tfrac{6}{3}, \tfrac{8}{4}, \tfrac{10}{5}, \tfrac{12}{6}, \text{\&c.}$$

each of them making two integers; for the numerator of each, divided by its denominator, gives 2. So all the fractions

$$\tfrac{3}{1}, \tfrac{6}{2}, \tfrac{9}{3}, \tfrac{12}{4}, \tfrac{15}{5}, \tfrac{18}{6}, \text{\&c.}$$

are equal to one another, since 3 is their common value.

86. We may likewise represent the value of any fraction in an infinite variety of ways. For if we multiply both the numerator and the denominator of a fraction by the same number, which may be assumed at pleasure, this fraction will still preserve the same value. For this reason, all the fractions

$$\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \tfrac{5}{10}, \tfrac{6}{12}, \tfrac{7}{14}, \tfrac{8}{16}, \tfrac{9}{18}, \tfrac{10}{20}, \text{\&c.}$$

are equal, the value of each being $\tfrac{1}{2}$. Also,

$$\tfrac{1}{3}, \tfrac{2}{6}, \tfrac{3}{9}, \tfrac{4}{12}, \tfrac{5}{15}, \tfrac{6}{18}, \tfrac{7}{21}, \tfrac{8}{24}, \tfrac{9}{27}, \tfrac{10}{30}, \text{\&c.}$$

are equal fractions, the value of each being $\tfrac{1}{3}$. The fractions

$$\tfrac{2}{3}, \tfrac{4}{6}, \tfrac{8}{12}, \tfrac{10}{15}, \tfrac{12}{18}, \tfrac{14}{21}, \tfrac{16}{24}, \text{\&c.}$$

have likewise all the same value. Hence we may conclude, in general, that the fraction $\frac{a}{b}$ may be represented by any of the following expressions, each of which is equal to $\frac{a}{b}$; viz.

merator, having 0 for its denominator, is, when expanded, precisely the same as $\tfrac{1}{0}$.

Thus, $\tfrac{2}{0} = \dfrac{2}{2-2}$, by division becomes

$$\begin{array}{l}
2-2)2 \quad (1+1+1, \text{\&c. ad infinitum} \\
\quad\ \ \underline{2-2} \\
\qquad\quad 2 \\
\qquad\quad \underline{2-2} \\
\qquad\qquad\ 2 \\
\qquad\qquad\ \underline{2-2} \\
\qquad\qquad\quad 2, \text{\&c.}
\end{array}$$

$$\frac{a}{b}, \frac{2a}{2b}, \frac{3a}{3b}, \frac{4a}{4b}, \frac{5a}{5b}, \frac{6a}{6b}, \frac{7a}{7b}, \text{\&c.}$$

87. To be convinced of this, we have only to write for the value of the fraction $\frac{a}{b}$ a certain letter c, representing by this letter c the quotient of the division of a by b; and to recollect that the multiplication of the quotient c by the divisor b must give the dividend. For since c multiplied by b gives a, it is evident that c multiplied by $2b$ will give $2a$, that c multiplied by $3b$ will give $3a$, and that, in general, c multiplied by mb will give ma. Now, changing this into an example of division, and dividing the product ma by mb, one of the factors, the quotient must be equal to the other factor c; but ma divided by mb gives also the fraction $\frac{ma}{mb}$, which is consequently equal to c; which is what was to be proved: for c having been assumed as the value of the fraction $\frac{a}{b}$, it is evident that this fraction is equal to the fraction $\frac{ma}{mb}$, whatever be the value of m.

88. We have seen that every fraction may be represented in an infinite number of forms, each of which contains the same value; and it is evident that of all these forms, that which is composed of the least numbers, will be most easily understood. For example, we might substitute, instead of $\frac{2}{3}$, the following fractions,

$$\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \text{\&c.}$$

but of all these expressions $\frac{2}{3}$ is that of which it is easiest to form an idea. Here therefore a problem arises, how a fraction, such as $\frac{8}{12}$, which is not expressed by the least possible numbers, may be reduced to its simplest form, or to *its least terms;* that is to say, in our present example, to $\frac{2}{3}$.

89. It will be easy to resolve this problem, if we consider that a fraction still preserves its value, when we multiply both its terms, or its numerator and denominator, by the same number. For from this it also follows, that if we divide the numerator and denominator of a fraction by the same number, the fraction will still preserve the same value. This is made more evident by means of the general expression $\frac{ma}{mb}$; for if we divide both the numerator ma and the denominator mb by the number m, we obtain the fraction $\frac{a}{b}$, which, as was before proved, is equal to $\frac{ma}{mb}$.

90. In order therefore to reduce a given fraction to its least terms, it is required to find a number, by which both the numerator and denominator may be divided. Such a number is called a *common divisor;* and as long as we can find a common divisor to the numerator and the denominator, it is certain that the fraction may be reduced to a lower form; but, on the contrary, when we see that, except unity, no other common divisor can be found, this shews that the fraction is already in its simplest form.

91. To make this more clear, let us consider the fraction $\frac{48}{120}$. We see immediately that both the terms are divisible by 2, and that there results the fraction $\frac{24}{60}$; which may also be divided by 2, and reduced to $\frac{12}{30}$; and as this likewise has 2 for a common divisor, it is evident that it may be reduced to $\frac{6}{15}$. But now we easily perceive, that the numerator and denominator are still divisible by 3; performing this division, therefore, we obtain the fraction $\frac{2}{5}$, which is equal to the fraction proposed, and gives the simplest expression to which it can be reduced; for 2 and 5 have no common divisor but 1, which cannot diminish these numbers any farther.

92. This property of fractions preserving an invariable value, whether we divide or multiply the numerator and denominator by the same number, is of the greatest importance, and is the principal foundation of the doctrine of fractions. For example, we can seldom add together two fractions, or subtract the one from the other, before we have, by means of this property, reduced them to other forms; that is to say, to expressions whose denominators are equal. Of this we shall treat in the following chapter.

93. We will conclude the present, however, by remarking, that all whole numbers may also be represented by fractions. For example, 6 is the same as $\frac{6}{1}$, because 6 divided by 1 makes 6; we may also, in the same manner, express the number 6 by the fractions $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$, $\frac{36}{6}$, and an infinite number of others, which have the same value.

QUESTIONS FOR PRACTICE.

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its lowest terms. *Ans.* $\frac{x}{a^2}$.

2. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms. *Ans.* $\frac{x^2-bx}{x+b}$.

3. Reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms. *Ans.* $\frac{x^2+b^2}{x^3}$.

4. Reduce $\frac{x^2-y^2}{x^4-y^4}$ to its lowest terms. *Ans.* $\frac{1}{x^2+y^2}$.

5. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Ans. $\frac{a^2+x^2}{a-x}$.

6. Reduce $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$ to its lowest terms.

Ans. $\frac{5a^4+5a^3x}{a^2x+ax^2+x^3}$

CHAP. IX.

Of the Addition *and* Subtraction *of* Fractions.

94. When fractions have equal denominators, there is no difficulty in adding and subtracting them; for $\frac{2}{7}+\frac{3}{7}$ is equal to $\frac{5}{7}$, and $\frac{4}{7}-\frac{2}{7}$ is equal to $\frac{2}{7}$. In this case, therefore, either for addition or subtraction, we alter only the numerators, and place the common denominator under the line, thus;

$\frac{7}{100}+\frac{9}{100}-\frac{12}{100}-\frac{15}{100}+\frac{20}{100}$ is equal to $\frac{9}{100}$;
$\frac{24}{50}-\frac{7}{50}-\frac{12}{50}+\frac{31}{50}$ is equal to $\frac{36}{50}$, or $\frac{18}{25}$;
$\frac{16}{20}-\frac{3}{20}-\frac{11}{20}+\frac{14}{20}$ is equal to $\frac{16}{20}$, or $\frac{4}{5}$;

also $\frac{1}{3}+\frac{2}{3}$ is equal to $\frac{3}{3}$, or 1, that is to say, an integer; and $\frac{2}{4}-\frac{3}{4}+\frac{1}{4}$ is equal to $\frac{0}{4}$, that is to say, nothing, or 0.

95. But when fractions have not equal denominators, we can always change them into other fractions that have the same denominator. For example, when it is proposed to add together the fractions $\frac{1}{2}$ and $\frac{1}{3}$, we must consider that $\frac{1}{2}$ is the same as $\frac{3}{6}$, and that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$; we have therefore, instead of the two fractions proposed, $\frac{3}{6}+\frac{2}{6}$, the sum of which is $\frac{5}{6}$. And if the two fractions were united by the sign *minus* as $\frac{1}{2}-\frac{1}{3}$, we should have $\frac{3}{6}-\frac{2}{6}$, or $\frac{1}{6}$.

As another example, let the fractions proposed be $\frac{3}{4}+\frac{5}{8}$. Here, since $\frac{3}{4}$ is the same as $\frac{6}{8}$, this value may be substituted for $\frac{3}{4}$, and we may then say $\frac{6}{8}+\frac{5}{8}$ makes $\frac{11}{8}$, or $1\frac{3}{8}$.

Suppose farther, that the sum of $\frac{1}{3}$ and $\frac{1}{4}$ were required, I say that it is $\frac{7}{12}$; for $\frac{1}{3}=\frac{4}{12}$, and $\frac{1}{4}=\frac{3}{12}$: therefore $\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$.

96. We may have a greater number of fractions to reduce