$log.\ 50 = 2 - x$, $log.\ 500 = 3 - x$
$log.\ 5000 = 4 - x$, $log.\ 50000 = 5 - x$, &c.
$log.\ 25 = 2 - 2x$, $log.\ 125 = 3 - 3x$
$log.\ 625 = 4 - 4x$, $log.\ 3125 = 5 - 5x$, &c.
$log.\ 250 = 3 - 2x$, $log.\ 2500 = 4 - 2x$
$log.\ 25000 = 5 - 2x$, $log.\ 250000 = 6 - 2x$, &c.
$log.\ 1250 = 4 - 3x$, $log.\ 12500 = 5 - 3x$
$log.\ 125000 = 6 - 3x$, $log.\ 1250000 = 7 - 3x$, &c.
$log.\ 6250 = 5 - 4x$, $log.\ 62500 = 6 - 4x$
$log.\ 625000 = 7 - 4x$, $log.\ 6250000 = 8 - 4x$, &c.
and so on.

240. If we knew the logarithm of 3, this would be the means also of determining a number of other logarithms; as appears from the following examples. Let the *log.* 3 be represented by the letter y: then,

$log.\ 30 = y + 1$, $log.\ 300 = y + 2$
$log.\ 3000 = y + 3$, $log.\ 30000 = y + 4$, &c.
$log.\ 9 = 2y$, $log.\ 27 = 3y$, $log.\ 81 = 4y$, &c. we shall have also,
$log.\ 6 = x + y$, $log.\ 12 = 2x + y$, $log.\ 18 = x + 2y$,
$log.\ 15 = log.\ 3 + log.\ 5 = y + 1 - x$.

241. We have already seen that all numbers arise from the multiplication of prime numbers. If therefore we only knew the logarithms of all the prime numbers, we could find the logarithms of all the other numbers by simple additions. The number 210, for example, being formed by the factors 2, 3, 5, 7, its logarithm will be *log.* 2 + *log.* 3 + *log.* 5 + *log.* 7. In the same manner, since $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$, we have $log.\ 360 = 3\ log.\ 2 + 2\ log.\ 3 + log.\ 5$. It is evident, therefore, that by means of the logarithms of the prime numbers, we may determine those of all others; and that we must first apply to the determination of the former, if we would construct Tables of Logarithms.

CHAP. XXIII.

Of the Method *of expressing* Logarithms.

242. We have seen that the logarithm of 2 is greater than $\frac{3}{10}$, and less than $\frac{1}{3}$, and that, consequently, the exponent of 10 must fall between those two fractions, in order that the power may become 2. Now, although we know this, yet

whatever fraction we assume on this condition, the power resulting from it will be always an irrational number, greater or less than 2; and, consequently, the logarithm of 2 cannot be accurately expressed by such a fraction: therefore we must content ourselves with determining the value of that logarithm by such an approximation as may render the error of little or no importance; for which purpose, we employ what are called *decimal fractions*, the nature and properties of which ought to be explained as clearly as possible.

243. It is well known that, in the ordinary way of writing numbers by means of the ten figures, or characters,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9,

the first figure on the right alone has its natural signification; that the figures in the second place have ten times the value which they would have had in the first; that the figures in the third place have a hundred times the value; and those in the fourth a thousand times, and so on: so that as they advance towards the left, they acquire a value ten times greater than they had in the preceding rank. Thus, in the number 1765, the figure 5 is in the first place on the right, and is just equal to 5; in the second place is 6; but this figure, instead of 6, represents 10×6, or 60: the figure 7 is in the third place, and represents 100×7, or 700; and lastly, the 1, which is in the fourth row, becomes 1000; so that we read the given number thus;

One thousand, seven hundred, and sixty-five.

244. As the value of figures becomes always ten times greater, as we go from the right towards the left, and as it consequently becomes continually ten times less as we go from the left towards the right; we may, in conformity with this law, advance still farther towards the right, and obtain figures whose value will continue to become ten times less than in the preceding place: but it must be observed, that the place where the figures have their natural value is marked by a point. So that if we meet, for example, with the number 36·54892, it is to be understood in this manner: the figure 6, in the first place, has its natural value; and the figure 3, which is in the second place to the left, means 30. But the figure 5, which comes after the point, expresses only $\frac{5}{10}$; and the 4 is equal only to $\frac{4}{100}$; the figure 8 is equal to $\frac{8}{1000}$; the figure 9 is equal to $\frac{9}{10000}$; and the figure 2 is equal to $\frac{2}{100000}$. We see then, that the more those figures advance towards the right, the more their

values diminish, and at last, those values become so small, that they may be considered as nothing *.

245. This is the kind of numbers which we call *decimal fractions*, and in this manner logarithms are represented in the Tables. The logarithm of 2, for example, is expressed by 0·3010300; in which we see, 1st. That since there is 0 before the point, this logarithm does not contain an integer; 2dly, that its value is $\frac{3}{10}+\frac{0}{100}+\frac{1}{1000}+\frac{0}{10000}+\frac{3}{100000}+\frac{0}{1000000}+\frac{0}{10000000}$. We might have left out the two last ciphers, but they serve to shew that the logarithm in question contains none of those parts, which have 1000000 and 10000000 for the denominator. It is however to be understood, that, by continuing the series, we might have found still smaller parts; but with regard to these, they are neglected, on account of their extreme minuteness.

246. The logarithm of 3 is expressed in the Table by 0·4771213; we see, therefore, that it contains no integer, and that it is composed of the following fractions: $\frac{4}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{1}{10000}+\frac{2}{100000}+\frac{1}{1000000}+\frac{3}{10000000}$. But we must not suppose that the logarithm is thus expressed with the utmost exactness; we are only certain that the error is less than $\frac{1}{10000000}$; which is certainly so small, that it may very well be neglected in most calculations.

247. According to this method of expressing logarithms, that of 1 must be represented by 0·0000000, since it is really $= 0$: the logarithm of 10 is 1·0000000, where it evidently is exactly $= 1$: the logarithm of 100 is 2·0000000, or 2. And hence we may conclude, that the logarithms of all numbers, which are included between 10 and 100, and

* The operations of arithmetic are performed with decimal fractions in the same manner nearly, as with whole numbers; some precautions only are necessary, after the operation, to place the point properly, which separates the whole numbers from the decimals. On this subject, we may consult almost any of the treatises on arithmetic. In the multiplication of these fractions, when the multiplicand and multiplier contain a great number of decimals, the operation would become too long, and would give the result much more exact than is for the most part necessary; but it may be simplified by a method, which is not to be found in many authors, and which is pointed out by M. Marie in his edition of the mathematical lessons of M. de la Caille, where he likewise explains a similar method for the division of decimals. F. T.

The method alluded to in this note is clearly explained in Bonnycastle's Arithmetic.

consequently composed of two figures, are comprehended between 1 and 2, and therefore must be expressed by 1 *plus* a decimal fraction, as *log*. 50 = 1·6989700; its value therefore is unity, *plus* $\frac{6}{10} + \frac{9}{100} + \frac{8}{1000} + \frac{9}{10000} + \frac{7}{100000}$: and it will be also easily perceived, that the logarithms of numbers, between 100 and 1000, are expressed by the integer 2 with a decimal fraction: those of numbers between 1000 and 10000, by 3 *plus* a decimal fraction: those of numbers between 10000 and 100000, by 4 integers *plus* a decimal fraction, and so on. Thus, the *log*. 800, for example, is 2·9030900; that of 2290 is 3·3598355, &c.

248. On the other hand, the logarithms of numbers which are less than 10, or expressed by a single figure, do not contain an integer, and for this reason we find 0 before the point: so that we have two parts to consider in a logarithm. First, that which precedes the point, or the integral part; and the other, the decimal fractions that are to be added to the former. The integral part of a logarithm, which is usually called the *characteristic*, is easily determined from what we have said in the preceding article. Thus, it is 0, for all the numbers which have but *one figure;* it is 1, for those which have *two;* it is 2, for those which have *three;* and, in general, it is always one less than the number of figures. If therefore the logarithm of 1766 be required, we already know that the first part, or that of the integers, is necessarily 3.

249. So reciprocally, we know at the first sight of the integer part of a logarithm, how many figures compose the number answering to that logarithm; since the number of those figures always exceed the integer part of the logarithm by unity. Suppose, for example, the number answering to the logarithm 6·4771213 were required, we know immediately that that number must have seven figures, and be greater than 1000000. And in fact this number is 3000000; for *log*. 3000000 = *log*. 3 + *log*. 1000000. Now *log*. 3 = 0·4771213, and *log*. 1000000 = 6, and the sum of those two logarithms is 6·4771213.

250. The principal consideration therefore with respect to each logarithm is, the decimal fraction which follows the point, and even that, when once known, serves for several numbers. In order to prove this, let us consider the logarithm of the number 365; its first part is undoubtedly 2; with respect to the other, or the decimal fraction, let us at present represent it by the letter x; we shall have *log*. 365 $= 2 + x$; then multiplying continually by 10, we shall

have $log.\ 3650 = 3 + x$; $log.\ 36500 = 4 + x$; $log.\ 365000 = 5 + x$, and so on.

But we can also go back, and continually divide by 10; which will give us $log.\ 36{\cdot}5 = 1 + x$; $log.\ 3{\cdot}65 = 0 + x$; $log.\ 0{\cdot}365 = -1 + x$; $log.\ 0{\cdot}0365 = -2 + x$; $log.\ 0{\cdot}00365 = -3 + x$, and so on.

251. All those numbers then which arise from the figures 365, whether preceded, or followed, by ciphers, have always the same decimal fraction for the second part of the logarithm: and the whole difference lies in the integer before the point, which, as we have seen, may become negative; namely, when the number proposed is less than 1. Now, as ordinary calculators find a difficulty in managing negative numbers, it is usual, in those cases, to increase the integers of the logarithm by 10, that is, to write 10 instead of 0 before the point; so that instead of -1 we have 9; instead of -2 we have 8; instead of -3 we have 7, &c.; but then we must remember, that the characteristic has been taken ten units too great, and by no means suppose that the number consists of 10, 9, or 8 figures. It is likewise easy to conceive, that, if in the case we speak of, this characteristic be less than 10, we must write the figures of the number after a point, to shew that they are decimals: for example, if the characteristic be 9, we must begin at the first place after a point; if it be 8, we must also place a cipher in the first row, and not begin to write the figures till the second: thus 9·5622929 would be the logarithm of 0·365, and 8·5622929 the log. of 0·0365. But this manner of writing logarithms is principally employed in Tables of sines.

252. In the common Tables, the decimals of logarithms are usually carried to seven places of figures, the last of which consequently represents the $\frac{1}{10000000}$ part, and we are sure that they are never erroneous by the whole of this part, and that therefore the error cannot be of any importance. There are, however, calculations in which we require still greater exactness; and then we employ the large Tables of Vlacq, where the logarithms are calculated to ten decimal places *.

* The most valuable set of tables we are acquainted with are those published by Dr. Hutton, late Professor of Mathematics at the Royal Military Academy, Woolwich, under the title of, "Mathematical Tables; containing common, hyperbolic, and logistic logarithms. Also sines, tangents, &c. to which is prefixed a large and original history of the discoveries and writings relating to those subjects."

253. As the first part, or characteristic of a logarithm, is subject to no difficulty, it is seldom expressed in the Tables; the second part only is written, or the seven figures of the decimal fraction. There is a set of English Tables in which we find the logarithms of all numbers from 1 to 100000, and even those of greater numbers; for small additional Tables shew what is to be added to the logarithms, in proportion to the figures, which the proposed numbers have more than those in the Tables. We easily find, for example, the logarithm of 379456, by means of that of 37945 and the small Tables of which we speak *.

254. From what has been said, it will easily be perceived, how we are to obtain from the Tables the number corresponding to any logarithm which may occur. Thus, in multiplying the numbers 343 and 2401; since we must add

* The English Tables spoken of in the text are those which were published by Sherwin in the beginning of the last century, and have been several times reprinted; they are likewise to be found in the tables of Gardener, which are commonly made use of by astronomers, and which have been reprinted at Avignon. With respect to these Tables it is proper to remark, that as they do not carry logarithms farther than seven places, independently of the characteristic, we cannot use them with perfect exactness except on numbers that do not exceed six digits; but when we employ the great Tables of Vlacq, which carry the logarithms as far as ten decimal places, we may, by taking the proportional parts, work, without error, upon numbers that have as many as nine digits. The reason of what we have said, and the method of employing these Tables in operations upon still greater numbers, is well explained in Saunderson's "Elements of Algebra," Book IX. Part II.

It is farther to be observed, that these Tables only give the logarithms answering to given numbers, so that when we wish to get the numbers answering to given logarithms, it is seldom that we find in the Tables the precise logarithms that are given, and we are for the most part under the necessity of seeking for these numbers in an indirect way, by the method of interpolation. In order to supply this defect, another set of Tables was published at London in 1742, under the title of "The Antilogarithmic Canon, &c. by James Dodson." He has arranged the decimals of logarithms from 0,0001 to 1,0000, and opposite to them, in order, the corresponding numbers carried as far as eleven places. He has likewise given the proportional parts necessary for determining the numbers, which answer to the intermediate logarithms that are not to be found in the Table. F. T.

together the logarithms of those numbers, the calculation will be as follows:

$$\left.\begin{array}{l} log.\ 343 = 2{\cdot}5352941 \\ log.\ 2401 = 3{\cdot}3803922 \end{array}\right\}\text{added}$$

$$\begin{array}{rl} \overline{\quad 5{\cdot}9156863} & \text{their sum} \\ log.\ 823540 = 5{\cdot}9156847 & \text{nearest tabular log.} \\ \overline{\qquad 16} & \text{difference,} \end{array}$$

which in the Table of Differences answers to 3; this therefore being used instead of the cipher, gives 823543 for the product sought: for the sum is the logarithm of the product required; and its characteristic 5 shews that the product is composed of 6 figures; which are found as above.

255. But it is in the extraction of roots that logarithms are of the greatest service; we shall therefore give an example of the manner in which they are used in calculations of this kind. Suppose, for example, it were required to extract the square root of 10. Here we have only to divide the logarithm of 10, which is 1·0000000 by 2; and the quotient 0·5000000 is the logarithm of the root required. Now, the number in the Tables which answers to that logarithm is 3·16228, the square of which is very nearly equal to 10, being only one hundred thousandth part too great *.

* In the same manner, we may extract any other root, by dividing the log. of the number by the denominator of the index of the root to be extracted; that is, to extract the cube root, divide the log. by 3, the fourth root by 4, and so on for any other extraction. For example, if the 5th root of 2 were required, the log. of 2 is 0·3010300: therefore

$$\begin{array}{l} 5)0{\cdot}3010300 \\ \overline{\quad 0{\cdot}0602060} \end{array}$$

is the log. of the root, which by the Tables is found to correspond to 1·1497; and hence we have $\sqrt[5]{2} = 1{\cdot}1497$. When the index, or characteristic of the log. is negative, and not divisible by the denominator of the index of the root to be extracted; then as many units must be borrowed as will make it exactly divisible, carrying those units to the next figure, as in common division.