rithm of c, that is, *log. c*, and next taking the half of that logarithm, or $\frac{1}{2}log. c$, we should have the logarithm of the square root required: we have therefore only to look in the Tables for the number answering to that logarithm, in order to obtain the root required.

230. We have already seen, that the numbers, 1, 2, 3, 4, 5, 6, &c. that is to say, all positive numbers, are logarithms of the root a, and of its positive powers; consequently, logarithms of numbers greater than unity: and, on the contrary, that the negative numbers, as -1, -2, &c. are logarithms of the fractions $\frac{1}{a}$, $\frac{1}{a^2}$, &c. which are less than unity, but yet greater than nothing.

Hence, it follows, that, if the logarithm be positive, the number is always greater than unity: but if the logarithm be negative, the number is always less than unity, and yet greater than 0; consequently, we cannot express the logarithms of negative numbers: we must therefore conclude, that the logarithms of negative numbers are impossible, and that they belong to the class of imaginary quantities.

231. In order to illustrate this more fully, it will be proper to fix on a determinate number for the root a. Let us make choice of that, on which the common Logarithmic Tables are formed, that is, the number 10, which has been preferred, because it is the foundation of our Arithmetic. But it is evident that any other number, provided it were greater than unity, would answer the same purpose: and the reason why we cannot suppose $a =$ unity, or 1, is manifest; because all the powers a^b would then be constantly equal to unity, and could never become equal to another given number, c.

CHAP. XXII.

Of the Logarithmic Tables *now in use.*

232. In those Tables, as we have already mentioned, we begin with the supposition, that the root a is $= 10$; so that the logarithm of any number, c, is the exponent to which we must raise the number 10, in order that the power resulting from it may be equal to the number c; or if we denote the logarithm of c by L.c, we shall always have $10^{L.c} = c$.

233. We have already observed, that the logarithm of the number 1 is always 0; and we have also $10^0 = 1$; consequently, *log.* $1 = 0$; *log.* $10 = 1$; *log.* $100 = 2$; *log.* $1000 = 3$; *log.* $10000 = 4$; *log.* $100000 = 5$; *log.* $1000000 = 6$. Farther, *log.* $\frac{1}{10} = -1$; *log.* $\frac{1}{100} = -2$; *log.* $\frac{1}{1000} = -3$; *log.* $\frac{1}{10000} = -4$; *log.* $\frac{1}{100000} = -5$; *log.* $\frac{1}{1000000} = -6$.

234. The logarithms of the principal numbers, therefore, are easily determined; but it is much more difficult to find the logarithms of all the other intervening numbers; and yet they must be inserted in the Tables. This however is not the place to lay down all the rules that are necessary for such an inquiry; we shall therefore at present content ourselves with a general view only of the subject.

235. First, since *log.* $1 = 0$, and *log.* $10 = 1$, it is evident that the logarithms of all numbers between 1 and 10 must be included between 0 and unity; and, consequently, be greater than 0, and less than 1. It will therefore be sufficient to consider the single number 2; the logarithm of which is certainly greater than 0, but less than unity: and if we represent this logarithm by the letter x, so that *log.* $2 = x$, the value of that letter must be such as to give exactly $10^x = 2$.

We easily perceive, also, that x must be considerably less than $\frac{1}{2}$, or which amounts to the same thing, $10^{\frac{1}{2}}$ is greater than 2; for if we square both sides, the square of $10^{\frac{1}{2}} = 10$, and the square of $2 = 4$. Now, this latter is much less than the former: and, in the same manner, we see that x is also less than $\frac{1}{3}$; that is to say, $10^{\frac{1}{3}}$ is greater than 2: for the cube of $10^{\frac{1}{3}}$ is 10, and that of 2 is only 8. But, on the contrary, by making $x = \frac{1}{4}$, we give it too small a value; because the fourth power of $10^{\frac{1}{4}}$ being 10, and that of 2 being 16, it is evident that $10^{\frac{1}{4}}$ is less than 2. Thus, we see that x, or the *log.* 2, is less than $\frac{1}{3}$, but greater than $\frac{1}{4}$: and, in the same manner, we may determine, with respect to every fraction contained between $\frac{1}{4}$ and $\frac{1}{3}$, whether it be too great or too small.

In making trial, for example, with $\frac{2}{7}$, which is less than $\frac{1}{3}$, and greater than $\frac{1}{4}$, 10^x, or $10^{\frac{2}{7}}$, ought to be $= 2$; or the seventh power of $10^{\frac{2}{7}}$, that is to say, 10^2, or 100, ought to be equal to the seventh power of 2, or 128; which is consequently greater than 100. We see, therefore, that $\frac{2}{7}$ is less than *log.* 2, and that *log.* 2, which was found less than $\frac{1}{3}$, is however greater than $\frac{2}{7}$.

Let us try another fraction, which, in consequence of what we have already found, must be contained between $\frac{2}{7}$ and $\frac{1}{3}$. Such a fraction between these limits is $\frac{3}{10}$; and it is therefore required to find whether $10^{\frac{3}{10}} = 2$; if this be the case, the tenth powers of those numbers are also equal: but the tenth power of $10^{\frac{3}{10}}$ is $10^3 = 1000$, and the tenth power of 2 is 1024; we conclude therefore, that $10^{\frac{3}{10}}$ is less than 2, and, consequently, that $\frac{3}{10}$ is too small a fraction, and therefore the *log*. 2, though less than $\frac{1}{3}$, is yet greater than $\frac{3}{10}$.

236. This discussion serves to prove, that *log*. 2 has a determinate value, since we know that it is certainly greater than $\frac{3}{10}$, but less than $\frac{1}{3}$; we shall not however proceed any farther in this investigation at present. Being therefore still ignorant of its true value, we shall represent it by x, so that *log*. $2 = x$; and endeavour to shew how, if it were known, we could deduce from it the logarithms of an infinity of other numbers. For this purpose, we shall make use of the equation already mentioned, namely, *log*. $cd =$ *log*. $c +$ *log*. d, which comprehends the property, that the logarithm of a product is found by adding together the logarithms of the factors.

237. First, as *log*. $2 = x$, and *log*. $10 = 1$, we shall have

log. $20 = x + 1$, *log*. $200 = x + 2$
log. $2000 = x + 3$, *log*. $20000 = x + 4$
log. $200000 = x + 5$, *log*. $2000000 = x + 6$, &c.

238. Farther, as *log*. $c^2 = 2$ *log*. c, and *log*. $c^3 = 3$ *log*. c, and *log*. $c^4 = 4$ *log*. c, &c. we have

log. $4 = 2x$; *log*. $8 = 3x$; *log*. $16 = 4x$; *log*. $32 = 5x$; *log*. $64 = 6x$, &c. Hence we find also, that

log. $40 = 2x + 1$, *log*. $400 = 2x + 2$
log. $4000 = 2x + 3$, *log*. $40000 = 2x + 4$, &c.
log. $80 = 3x + 1$, *log*. $800 = 3x + 2$
log. $8000 = 3x = 3$, *log*. $80000 = 3x + 4$, &c.
log. $160 = 4x + 1$, *log*. $1600 = 4x + 2$
log. $16000 = 4x + 3$, *log*. $160000 = 4x + 4$, &c.

239. Let us resume also the other fundamental equation, *log*. $\frac{c}{d} =$ *log*. $c -$ *log*. d, and let us suppose $c = 10$, and $d = 2$; since *log*. $10 = 1$, and *log*. $2 = x$, we shall have *log*. $\frac{10}{2}$, or *log*. $5 = 1 - x$, and shall deduce from hence the following equations:

$log.\ 50 = 2 - x,$ $log.\ 500 = 3 - x$
$log.\ 5000 = 4 - x,$ $log.\ 50000 = 5 - x,$ &c.
$log.\ 25 = 2 - 2x,$ $log.\ 125 = 3 - 3x$
$log.\ 625 = 4 - 4x,$ $log.\ 3125 = 5 - 5x,$ &c.
$log.\ 250 = 3 - 2x,$ $log.\ 2500 = 4 - 2x$
$log.\ 25000 = 5 - 2x,$ $log.\ 250000 = 6 - 2x,$ &c.
$log.\ 1250 = 4 - 3x,$ $log.\ 12500 = 5 - 3x$
$log.\ 125000 = 6 - 3x,$ $log.\ 1250000 = 7 - 3x,$ &c.
$log.\ 6250 = 5 - 4x,$ $log.\ 62500 = 6 - 4x$
$log.\ 625000 = 7 - 4x,$ $log.\ 6250000 = 8 - 4x,$ &c.
and so on.

240. If we knew the logarithm of 3, this would be the means also of determining a number of other logarithms; as appears from the following examples. Let the *log.* 3 be represented by the letter y: then,

$log.\ 30 = y + 1,$ $log.\ 300 = y + 2$
$log.\ 3000 = y + 3,$ $log.\ 30000 = y + 4,$ &c.
$log.\ 9 = 2y,$ $log.\ 27 = 3y,$ $log.\ 81 = 4y,$ &c. we shall have also,
$log.\ 6 = x + y,$ $log.\ 12 = 2x + y,$ $log.\ 18 = x + 2y,$
$log.\ 15 = log.\ 3 + log.\ 5 = y + 1 - x.$

241. We have already seen that all numbers arise from the multiplication of prime numbers. If therefore we only knew the logarithms of all the prime numbers, we could find the logarithms of all the other numbers by simple additions. The number 210, for example, being formed by the factors 2, 3, 5, 7, its logarithm will be *log.* 2 + *log.* 3 + *log.* 5 + *log.* 7. In the same manner, since $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$, we have $log.\ 360 = 3\ log.\ 2 + 2\ log.\ 3 + log.\ 5$. It is evident, therefore, that by means of the logarithms of the prime numbers, we may determine those of all others; and that we must first apply to the determination of the former, if we would construct Tables of Logarithms.

CHAP. XXIII.

Of the Method *of expressing* Logarithms.

242. We have seen that the logarithm of 2 is greater than $\frac{3}{10}$, and less than $\frac{1}{3}$, and that, consequently, the exponent of 10 must fall between those two fractions, in order that the power may become 2. Now, although we know this, yet