$b = 3$ and $a^3 = c$, we know that the cube of a must be equal to the given number c, and consequently that $a = \sqrt[3]{c}$. It is therefore easy to conclude, generally, from this, how to determine the letter a by means of the letters c and b; for we must necessarily have $a = \sqrt[b]{c}$.

218. We have already remarked also the consequence which follows, when the given number is not a real power; a case which very frequently occurs; namely, that then the required root, a, can neither be expressed by integers, nor by fractions; yet since this root must necessarily have a determinate value, the same consideration led us to a new kind of numbers, which, as we observed, are called *surds*, or *irrational* numbers; and which we have seen are divisible into an infinite number of different sorts, on account of the great variety of roots. Lastly, by the same inquiry, we were led to the knowledge of another particular kind of numbers, which have been called *imaginary numbers*.

219. It remains now to consider the second question, which was to determine the exponent; the power c, and the root a, both being known. On this question, which has not yet occurred, is founded the important theory of Logarithms, the use of which is so extensive through the whole compass of mathematics, that scarcely any long calculation can be carried on without their assistance; and we shall find, in the following chapter, for which we reserve this theory, that it will lead us to another kind of numbers entirely new, as they cannot be ranked among the irrational numbers before mentioned.

CHAP. XXI.

Of Logarithms *in general.*

220. Resuming the equation $a^b = c$, we shall begin by remarking that, in the doctrine of Logarithms, we assume for the root a, a certain number taken at pleasure, and suppose this root to preserve invariably its assumed value. This being laid down, we take the exponent b such, that the power a^b becomes equal to a given number c; in which case this exponent b is said to be the *logarithm* of the number c. To express this, we shall use the letter L. or the initial letters *log*. Thus, by $b =$ L. c, or $b = log.\ c$,

we mean that b is equal to the logarithm of the number c, or that the logarithm of c is b.

221. We see then, that the value of the root a being once established, the logarithm of any number, c, is nothing more than the exponent of that power of a, which is equal to c: so that c being $= a^b$, b is the logarithm of the power a^b. If, for the present, we suppose $b = 1$, we have 1 for the logarithm of a^1, and consequently $log.\ a = 1$; but if we suppose $b = 2$, we have 2 for the logarithm of a^2; that is to say, $log.\ a^2 = 2$, and we may, in the same manner, obtain $log.\ a^3 = 3$; $log.\ a^4 = 4$; $log.\ a^5 = 5$, and so on.

222. If we make $b = 0$, it is evident that 0 will be the logarithm of a^0; but $a^0 = 1$; consequently $log.\ 1 = 0$, whatever be the value of the root a.

Suppose $b = -1$, then -1 will be the logarithm of a^{-1}; but $a^{-1} = \frac{1}{a}$; so that we have $log.\ \frac{1}{a} = -1$, and in the same manner, we shall have $log.\ \frac{1}{a^2} = -2$; $log.\ \frac{1}{a^3} = -3$; $log.\ \frac{1}{a^4} = -4$, &c.

223. It is evident, then, how we may represent the logarithms of all the powers of a, and even those of fractions, which have unity for the numerator, and for the denominator a power of a. We see also, that in all those cases the logarithms are integers; but it must be observed, that if b were a fraction, it would be the logarithm of an irrational number: if we suppose, for example, $b = \frac{1}{2}$, it follows, that $\frac{1}{2}$ is the logarithm of $a^{\frac{1}{2}}$, or of $\sqrt{a}$; consequently we have also $log.\ \sqrt{a} = \frac{1}{2}$; and we shall find, in the same manner, that $log.\ \sqrt[3]{a} = \frac{1}{3}$, $log.\ \sqrt[4]{a} = \frac{1}{4}$, &c.

224. But if it be required to find the logarithm of another number c, it will be readily perceived, that it can neither be an integer, nor a fraction; yet there must be such an exponent b, that the power a^b may become equal to the number proposed; we have therefore $b = log.\ c$; and generally, $a^{L.c} = c$.

225. Let us now consider another number d, whose logarithm has been represented in a similar manner by $log.\ d$; so that $a^{L.d} = d$. Here if we multiply this expression by the preceding one $a^{L.c} = c$, we shall have $a^{L.c+L.d} = cd$; hence, *the exponent is always the logarithm of the power;* consequently, $log.\ c + log.\ d = log.\ cd$. But if, instead of multiplying, we divide the former expression by the latter,

we shall obtain $a^{L.c-L.d} = \frac{c}{d}$; and, consequently, $log.\ c - log.\ d = log.\frac{c}{d}$.

226. This leads us to the two principal properties of logarithms, which are contained in the equations $log.\ c + log.\ d = log.\ cd$, and $log.\ c - log.\ d = log.\frac{c}{d}$. The former of these equations teaches us, that the logarithm of a product, as cd, is found by adding together the logarithms of the factors; and the latter shews us this property, namely, that the logarithm of a fraction may be determined by subtracting the logarithm of the denominator from that of the numerator.

227. It also follows from this, that when it is required to multiply, or divide, two numbers by one another, we have only to add, or subtract, their logarithms; and this is what constitutes the singular utility of logarithms in calculation: for it is evidently much easier to add, or subtract, than to multiply, or divide, particularly when the question involves large numbers.

228. Logarithms are attended with still greater advantages, in the involution of powers, and in the extraction of roots; for if $d = c$, we have, by the first property, $log.\ c + log.\ c = log.\ cc$, or c^2; consequently, $log.\ cc = 2\ log.\ c$; and, in the same manner, we obtain $log.\ c^3 = 3\ log.\ c$; $log.\ c^4 = 4\ log.\ c$; and, generally, $log.\ c^n = n\ log.\ c$. If we now substitute fractional numbers for n, we shall have, for example, $log.\ c^{\frac{1}{2}}$, that is to say, $log.\ \surd c, = \frac{1}{2} log.\ c$; and lastly, if we suppose n to represent negative numbers, we shall have $log.\ c^{-1}$, or $log.\ \frac{1}{c}, = -\ log.\ c$; $log.\ c^{-2}$, or $log.\ \frac{1}{c^2}, = -2 log.\ c$, and so on; which follows not only from the equation $log.\ c^n = n\ log.\ c$, but also from $log.\ 1 = 0$, as we have already seen.

229. If therefore we had Tables, in which logarithms were calculated for all numbers, we might certainly derive from them very great assistance in performing the most prolix calculations; such, for instance, as require frequent multiplications, divisions, involutions, and extractions of roots: for, in such Tables, we should have not only the logarithms of all numbers, but also the numbers answering to all logarithms. If it were required, for example, to find the square root of the number c, we must first find the loga-

rithm of c, that is, *log.* c, and next taking the half of that logarithm, or $\frac{1}{2}log.\ c$, we should have the logarithm of the square root required: we have therefore only to look in the Tables for the number answering to that logarithm, in order to obtain the root required.

230. We have already seen, that the numbers, 1, 2, 3, 4, 5, 6, &c. that is to say, all positive numbers, are logarithms of the root a, and of its positive powers; consequently, logarithms of numbers greater than unity: and, on the contrary, that the negative numbers, as -1, -2, &c. are logarithms of the fractions $\frac{1}{a}$, $\frac{1}{a^2}$, &c. which are less than unity, but yet greater than nothing.

Hence, it follows, that, if the logarithm be positive, the number is always greater than unity: but if the logarithm be negative, the number is always less than unity, and yet greater than 0; consequently, we cannot express the logarithms of negative numbers: we must therefore conclude, that the logarithms of negative numbers are impossible, and that they belong to the class of imaginary quantities.

231. In order to illustrate this more fully, it will be proper to fix on a determinate number for the root a. Let us make choice of that, on which the common Logarithmic Tables are formed, that is, the number 10, which has been preferred, because it is the foundation of our Arithmetic. But it is evident that any other number, provided it were greater than unity, would answer the same purpose: and the reason why we cannot suppose $a =$ unity, or 1, is manifest; because all the powers a^b would then be constantly equal to unity, and could never become equal to another given number, c.

CHAP. XXII.

Of the Logarithmic Tables *now in use.*

232. In those Tables, as we have already mentioned, we begin with the supposition, that the root a is $= 10$; so that the logarithm of any number, c, is the exponent to which we must raise the number 10, in order that the power resulting from it may be equal to the number c; or if we denote the logarithm of c by L.c, we shall always have $10^{L.c} = c$.