26. What multiplier will render $a + \sqrt{3}$ rational?
Ans. $a - \sqrt{3}$.

27. What multiplier will render $\sqrt{a} - \sqrt{b}$ rational?
Ans. $\sqrt{a} + \sqrt{b}$.

28. What multiplier will render the denominator of the fraction $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$ rational? *Ans.* $\sqrt{7} - \sqrt{3}$.

CHAP. XX.

Of the different Methods *of* Calculation, *and of their mutual* Connexion.

206. Hitherto we have only explained the different methods of calculation: namely, addition, subtraction, multiplication, and division; the involution of powers, and the extraction of roots. It will not be improper, therefore, in this place, to trace back the origin of these different methods, and to explain the connexion which subsists among them; in order that we may satisfy ourselves whether it be possible or not for other operations of the same kind to exist. This inquiry will throw new light on the subjects which we have considered.

In prosecuting this design, we shall make use of a new character, which may be employed instead of the expression that has been so often repeated, *is equal to;* this sign is $=$, which is read *is equal to:* thus, when I write $a = b$, this means that a is equal to b: so, for example, $3 \times 5 = 15$.

207. The first mode of calculation that presents itself to the mind, is undoubtedly addition, by which we add two numbers together and find their sum: let therefore a and b be the two given numbers, and let their sum be expressed by the letter c, then we shall have $a + b = c$; so that when we know the two numbers a and b, addition teaches us to find the number c.

208. Preserving this comparison $a + b = c$, let us reverse the question by asking, how we are to find the number b, when we know the numbers a and c.

It is here required therefore to know what number must be added to a, in order that the sum may be the number c: suppose, for example, $a = 3$ and $c = 8$; so that we must have $3 + b = 8$; then b will evidently be found by subtracting 3 from 8: and, in general, to find b, we must subtract a from c, whence arises $b = c - a$; for by adding a to both sides again, we have $b + a = c - a + a$, that is to say, $= c$, as we supposed.

209. Subtraction therefore takes place, when we invert the question which gives rise to addition. But the number which it is required to subtract may happen to be greater than that from which it is to be subtracted; as, for example, if it were required to subtract 9 from 5: this instance therefore furnishes us with the idea of a new kind of numbers, which we call negative numbers, because $5 - 9 = -4$.

210. When several numbers are to be added together, which are all equal, their sum is found by multiplication, and is called a product. Thus, ab means the product arising from the multiplication of a by b, or from the addition of the number a, b number of times; and if we represent this product by the letter c, we shall have $ab = c$; thus multiplication teaches us how to determine the number c, when the numbers a and b are known.

211. Let us now propose the following question: the numbers a and c being known, to find the number b. Suppose, for example, $a = 3$, and $c = 15$; so that $3b = 15$, and let us inquire by what number 3 must be multiplied, in order that the product may be 15; for the question proposed is reduced to this. This is a case of division; and the number required is found by dividing 15 by 3; and, in general, the number b is found by dividing c by a; from which results the equation $b = \frac{c}{a}$.

212. Now, as it frequently happens that the number c cannot be really divided by the number a, while the letter b must however have a determinate value, another new kind of numbers present themselves, which are called *fractions*. For example, suppose $a = 4$, and $c = 3$, so that $4b = 3$; then it is evident that b cannot be an integer, but a fraction, and that we shall have $b = \frac{3}{4}$.

213. We have seen that multiplication arises from addition; that is to say, from the addition of several equal quantities: and if we now proceed farther, we shall perceive that, from the multiplication of several equal quantities together, powers are derived; which powers are represented in a general manner by the expression a^b. This signifies that the number a must be multiplied as many times by itself, *minus* 1, as is indicated by the number b. And we know from what has been already said, that, in the present in-

stance, a is called the root, b the exponent, and a^b the power.

214. Farther, if we represent this power also by the letter c, we have $a^b = c$, an equation in which three letters a, b, c, are found; and we have shewn in treating of powers, how to find the power itself, that is, the letter c, when a root a and its exponent b are given. Suppose, for example, $a = 5$, and $b = 3$, so that $c = 5^3$: then it is evident that we must take the third power of 5, which is 125, so that in this case $c = 125$.

215. We have now seen how to determine the power c, by means of the root a and the exponent b; but if we wish to reverse the question, we shall find that this may be done in two ways, and that there are two different cases to be considered: for if two of these three numbers a, b, c, were given, and it were required to find the third, we should immediately perceive that this question would admit of three different suppositions, and consequently of three solutions. We have considered the case in which a and b were the given numbers, we may therefore suppose farther that c and a, or c and b, are known, and that it is required to determine the third letter. But, before we proceed any farther, let us point out a very essential distinction between involution and the two operations which lead to it. When, in addition, we reversed the question, it could be done only in one way; it was a matter of indifference whether we took c and a, or c and b, for the given numbers, because we might indifferently write $a + b$, or $b + a$; and it was also the same with multiplication; we could at pleasure take the letters a and b for each other, the equation $ab = c$ being exactly the same as $ba = c$: but in the calculation of powers, the same thing does not take place, and we can by no means write b^a instead of a^b; as a single example will be sufficient to illustrate: for let $a = 5$, and $b = 3$; then we shall have $a^b = 5^3 = 125$; but $b^a = 3^5 = 243$: which are two very different results.

216. It is evident then, that we may propose two questions more: one, to find the root a by means of the given power c, and the exponent b; the other, to find the exponent b, supposing the power c and the root a to be known.

217. It may be said, indeed, that the former of these questions has been resolved in the chapter on the extraction of roots; since if $b = 2$, for example, and $a^2 = c$, we know by this means, that a is a number whose square is equal to c, and consequently that $a = \sqrt{c}$. In the same manner, if

$b = 3$ and $a^3 = c$, we know that the cube of a must be equal to the given number c, and consequently that $a = \sqrt[3]{c}$. It is therefore easy to conclude, generally, from this, how to determine the letter a by means of the letters c and b; for we must necessarily have $a = \sqrt[b]{c}$.

218. We have already remarked also the consequence which follows, when the given number is not a real power; a case which very frequently occurs; namely, that then the required root, a, can neither be expressed by integers, nor by fractions; yet since this root must necessarily have a determinate value, the same consideration led us to a new kind of numbers, which, as we observed, are called *surds*, or *irrational* numbers; and which we have seen are divisible into an infinite number of different sorts, on account of the great variety of roots. Lastly, by the same inquiry, we were led to the knowledge of another particular kind of numbers, which have been called *imaginary numbers*.

219. It remains now to consider the second question, which was to determine the exponent; the power c, and the root a, both being known. On this question, which has not yet occurred, is founded the important theory of Logarithms, the use of which is so extensive through the whole compass of mathematics, that scarcely any long calculation can be carried on without their assistance; and we shall find, in the following chapter, for which we reserve this theory, that it will lead us to another kind of numbers entirely new, as they cannot be ranked among the irrational numbers before mentioned.

CHAP. XXI.

Of Logarithms *in general.*

220. Resuming the equation $a^b = c$, we shall begin by remarking that, in the doctrine of Logarithms, we assume for the root a, a certain number taken at pleasure, and suppose this root to preserve invariably its assumed value. This being laid down, we take the exponent b such, that the power a^b becomes equal to a given number c; in which case this exponent b is said to be the *logarithm* of the number c. To express this, we shall use the letter L. or the initial letters *log*. Thus, by $b =$ L. c, or $b =$ *log*. c,