CHAP. XIX.

Of the Method of representing Irrational Numbers *by* Fractional Exponents.

195. We have shewn in the preceding chapter, that the square of any power is found by doubling the exponent of that power; or that, in general, the square, or the second power, of a^n, is a^{2n}; and the converse also follows, viz. that the square root of the power a^{2n} is a^n, which is found by taking half the exponent of that power, or dividing it by 2.

196. Thus, the square root of a^2 is a^1, or a; that of a^4 is a^2; that of a^6 is a^3; and so on: and, as this is general, the square root of a^3 must necessarily be $a^{\frac{3}{2}}$, and that of a^5 must be $a^{\frac{5}{2}}$; consequently, we shall in the same manner have $a^{\frac{1}{2}}$ for the square root of a^1. Whence we see that $a^{\frac{1}{2}}$ is equal to $\sqrt{a}$; which new method of representing the square root demands particular attention.

197. We have also shewn, that, to find the cube of a power, as a^n, we must multiply its exponent by 3, and consequently that cube is a^{3n}.

Hence, conversely, when it is required to find the third, or cube root, of the power a^{3n}, we have only to divide that exponent by 3, and may therefore with certainty conclude, that the root required is a: consequently a^1, or a, is the cube root of a^3; a^2 is the cube root of a^6; a^3 of a^9; and so on.

198. There is nothing to prevent us from applying the same reasoning to those cases, in which the exponent is not divisible by 3, or from concluding that the cube root of a^2 is $a^{\frac{2}{3}}$, and that the cube root of a^4 is $a^{\frac{4}{3}}$, or $a^{1\frac{1}{3}}$; consequently, the third, or cube root of a, or a^1, must be $a^{\frac{1}{3}}$: whence also, it appears, that $a^{\frac{1}{3}}$ is the same as $\sqrt[3]{a}$.

199. It is the same with roots of a higher degree: thus, the fourth root of a will be $a^{\frac{1}{4}}$, which expression has the same value as $\sqrt[4]{a}$; the fifth root of a will be $a^{\frac{1}{5}}$, which is consequently equivalent to $\sqrt[5]{a}$; and the same observation may be extended to all roots of a higher degree.

200. We may therefore entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have just explained: but as we have been long accustomed to those signs, and meet with them in most books of Algebra, it might be wrong to banish them entirely from calculation; there is, however, sufficient reason also to employ, as is now frequently done, the other method of notation, because it manifestly corresponds with the nature of the thing. In fact, we see immediately that $a^{\frac{1}{2}}$ is the square root of a, because we know that the square of $a^{\frac{1}{2}}$, that is to say, $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{2}}$, is equal to a^1, or a.

201. What has been now said is sufficient to shew how we are to understand all other fractional exponents that may occur. If we have, for example, $a^{\frac{4}{3}}$, this means, that we must first take the fourth power of a, and then extract its cube, or third root; so that $a^{\frac{4}{3}}$ is the same as the common expression $\sqrt[3]{a^4}$. Hence, to find the value of $a^{\frac{3}{4}}$, we must first take the cube, or the third power of a, which is a^3, and then extract the fourth root of that power; so that $a^{\frac{3}{4}}$ is the same as $\sqrt[4]{a^3}$, and $a^{\frac{4}{5}}$ is equal to $\sqrt[5]{a^4}$, &c.

202. When the fraction which represents the exponent exceeds unity, we may express the value of the given quantity in another way: for instance, suppose it to be $a^{\frac{5}{2}}$; this quantity is equivalent to $a^{2\frac{1}{2}}$, which is the product of a^2 by $a^{\frac{1}{2}}$: now $a^{\frac{1}{2}}$ being equal to $\sqrt{a}$, it is evident that $a^{\frac{5}{2}}$ is equal to $a^2\sqrt{a^5}$: also $a^{\frac{10}{3}}$, or $a^{3\frac{1}{3}}$, is equal to $a^3\sqrt[3]{a}$; and $a^{\frac{15}{4}}$, that is, $a^{3\frac{3}{4}}$, expresses $a^3\sqrt[4]{a^3}$. These examples are sufficient to illustrate the great utility of fractional exponents.

203. Their use extends also to fractional numbers: for if there be given $\frac{1}{\sqrt{a}}$, we know that this quantity is equal to $\frac{1}{a^{\frac{1}{2}}}$; and we have seen already that a fraction of the form $\frac{1}{a^n}$ may be expressed by a^{-n}; so that instead of $\frac{1}{\sqrt{a}}$ we may use the expresssion $a^{-\frac{1}{2}}$; and, in the same man-

ner, $\frac{1}{\sqrt[3]{a}}$ is equal to $a^{-\frac{1}{3}}$. Again, if the quantity $\frac{a^2}{\sqrt[4]{a^3}}$ be proposed; let it be transformed into this, $\frac{a^2}{a^{\frac{3}{4}}}$, which is the product of a^2 by $a^{-\frac{3}{4}}$; now this product is equivalent to $a^{\frac{5}{4}}$, or to $a^{1\frac{1}{4}}$, or lastly, to $a\sqrt[4]{a}$. Practice will render similar reductions easy.

204. We shall observe, in the last place, that each root may be represented in a variety of ways; for $\sqrt{a}$ being the same as $a^{\frac{1}{2}}$, and $\frac{1}{2}$ being transformable into the fractions, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, &c. it is evident that $\sqrt{a}$ is equal to $\sqrt[4]{a^2}$, or to $\sqrt[6]{a^3}$, or to $\sqrt[8]{a^4}$, and so on. In the same manner, $\sqrt[3]{a}$, which is equal to $a^{\frac{1}{3}}$, will be equal to $\sqrt[6]{a^2}$, or to $\sqrt[9]{a^3}$, or to $\sqrt[12]{a^4}$. Hence also we see that the number a, or a^1, might be represented by the following radical expressions:

$$\sqrt[2]{a^2},\ \sqrt[3]{a^3},\ \sqrt[4]{a^4},\ \sqrt[5]{a^5},\ \&c.$$

205. This property is of great use in multiplication and division; for if we have, for example, to multiply $\sqrt[2]{a}$ by $\sqrt[3]{a}$, we write $\sqrt[6]{a^3}$ for $\sqrt[2]{a}$, and $\sqrt[6]{a^2}$ instead of $\sqrt[3]{a}$; so that in this manner we obtain the same radical sign for both, and the multiplication being now performed, gives the product $\sqrt[6]{a^5}$. The same result is also deduced from $a^{\frac{1}{2}+\frac{1}{3}}$, which is the product of $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{3}}$; for $\frac{1}{2}+\frac{1}{3}$ is $\frac{5}{6}$, and consequently the product required is $a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$.

On the contrary, if it were required to divide $\sqrt[2]{a}$, or $a^{\frac{1}{2}}$, by $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, we should have for the quotient $a^{\frac{1}{2}-\frac{1}{3}}$, or $a^{\frac{3}{6}-\frac{2}{6}}$, that is to say, $a^{\frac{1}{6}}$, or $\sqrt[6]{a}$.

QUESTIONS FOR PRACTICE RESPECTING SURDS.

1. Reduce 6 to the form of $\sqrt{5}$. *Ans.* $\sqrt{36}$.
2. Reduce $a+b$ to the form of $\sqrt{bc}$.
   Ans. $\sqrt{(aa+2ab+bb)}$.
3. Reduce $\frac{a}{b\sqrt{c}}$ to the form of $\sqrt{d}$. *Ans.* $\sqrt{\frac{aa}{bbc}}$.
4. Reduce a^2 and $b^{\frac{3}{2}}$ to the common index $\frac{1}{3}$.
   Ans. $\overline{a^6}|^{\frac{1}{3}}$, and $\overline{b^{\frac{9}{2}}}|^{\frac{1}{3}}$.

5. Reduce $\sqrt{48}$ to its simplest form. *Ans.* $4\sqrt{3}$.

6. Reduce $\sqrt{(a^3x - a^2x^2)}$ to its simplest form. *Ans.* $a\sqrt{(ax-xx^2)}$.

7. Reduce $\sqrt[3]{\frac{27a^3b^3}{8b-8a}}$ to its simplest form. *Ans.* $\frac{3ab}{2}\sqrt[3]{\frac{a}{b-a}}$.

8. Add $\sqrt{6}$ to $2\sqrt{6}$; and $\sqrt{8}$ to $\sqrt{50}$. *Ans.* $3\sqrt{6}$; and $7\sqrt{2}$.

9. Add $\sqrt{4a}$ and $\sqrt[4]{a^6}$ together. *Ans.* $(a+2)\sqrt{a}$.

10. Add $\left.\overline{\frac{b}{c}}\right|^{\frac{1}{2}}$ and $\left.\overline{\frac{c}{b}}\right|^{\frac{3}{2}}$ together. *Ans.* $\frac{b^2+c^2}{b\sqrt{bc}}$.

11. Subtract $\sqrt{4a}$ from $\sqrt[4]{a^6}$. *Ans.* $(a-2)\sqrt{a}$.

12. Subtract $\left.\overline{\frac{c}{b}}\right|^{\frac{3}{2}}$ from $\left.\overline{\frac{b}{c}}\right|^{\frac{1}{2}}$. *Ans.* $\frac{b^2-c^2}{b}\sqrt{\frac{1}{bc}}$.

13. Multiply $\sqrt{\frac{2ab}{3c}}$ by $\sqrt{\frac{9ad}{2b}}$. *Ans.* $\frac{3a^2d}{c}$.

14. Multiply $\sqrt{d}$ by $\sqrt[3]{ab}$. *Ans.* $\sqrt[6]{(a^2b^2d^3)}$.

15. Multiply $\sqrt{(4a - 3x)}$ by $2a$. *Ans.* $\sqrt{(16a^3 - 12a^2x)}$.

16. Multiply $\frac{a}{2b}\sqrt{(a - x)}$ by $(c - d)\sqrt{ax}$. *Ans.* $\frac{ac-ad}{2b}\sqrt{(a^2x-ax^2)}$.

17. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{4}}$; and $a^{\frac{1}{n}}$ by $a^{\frac{1}{m}}$. *Ans.* $a^{\frac{5}{12}}$; and $a^{\frac{m-n}{mn}}$.

18. Divide $\frac{ac-ad}{2b}\sqrt{(a^2x - ax^2)}$ by $\frac{a}{2b}\sqrt{(a - x)}$. *Ans.* $(c - d)\sqrt{ax}$.

19. Divide $a^2 - ad - b + d\sqrt{b}$ by $a - \sqrt{b}$. *Ans.* $a + \sqrt{b} - d$.

20. What is the cube of $\sqrt{2}$? *Ans.* $\sqrt{8}$.

21. What is the square of $3\sqrt[3]{bc^2}$? *Ans.* $9c\sqrt[3]{b^2c}$.

22. What is the fourth power of $\frac{a}{2b}\sqrt{\frac{2a}{c-b}}$? *Ans.* $\frac{a^6}{4b^4(c^2-2bc+b^2)}$.

23. What is the square of $3+\sqrt{5}$? *Ans.* $14+6\sqrt{5}$.

24. What is the square root of a^3? *Ans.* $a^{\frac{3}{2}}$; or $\sqrt{a^3}$.

25. What is the cube root of $\sqrt{(a^2 - x^2)}$? *Ans.* $\sqrt[6]{(a^2 - x^2)}$.

26. What multiplier will render $a + \surd 3$ rational?
Ans. $a - \surd 3$.

27. What multiplier will render $\surd a - \surd b$ rational?
Ans. $\surd a + \surd b$.

28. What multiplier will render the denominator of the fraction $\frac{\surd 6}{\surd 7 + \surd 3}$ rational? *Ans.* $\surd 7 - \surd 3$.

CHAP. XX.

Of the different Methods *of* Calculation, *and of their mutual* Connexion.

206. Hitherto we have only explained the different methods of calculation: namely, addition, subtraction, multiplication, and division; the involution of powers, and the extraction of roots. It will not be improper, therefore, in this place, to trace back the origin of these different methods, and to explain the connexion which subsists among them; in order that we may satisfy ourselves whether it be possible or not for other operations of the same kind to exist. This inquiry will throw new light on the subjects which we have considered.

In prosecuting this design, we shall make use of a new character, which may be employed instead of the expression that has been so often repeated, *is equal to;* this sign is $=$, which is read *is equal to:* thus, when I write $a = b$, this means that a is equal to b: so, for example, $3 \times 5 = 15$.

207. The first mode of calculation that presents itself to the mind, is undoubtedly addition, by which we add two numbers together and find their sum: let therefore a and b be the two given numbers, and let their sum be expressed by the letter c, then we shall have $a + b = c$; so that when we know the two numbers a and b, addition teaches us to find the number c.

208. Preserving this comparison $a + b = c$, let us reverse the question by asking, how we are to find the number b, when we know the numbers a and c.

It is here required therefore to know what number must be added to a, in order that the sum may be the number c: suppose, for example, $a = 3$ and $c = 8$; so that we must have $3 + b = 8$; then b will evidently be found by sub-