evident that if the root have the sign $+$, that is to say, if it be a positive number, its square must necessarily be a positive number also, because $+$ multiplied by $+$ makes $+$: hence the square of $+a$ will be $+aa$: but if the root be a negative number, as $-a$, the square is still positive, for it is $+aa$. We may therefore conclude, that $+aa$ is the square both of $+a$ and of $-a$, and that consequently every square has two roots, one positive, and the other negative. The square root of 25, for example, is both $+5$ and -5, because -5 multiplied by -5 gives 25, as well as $+5$ by $+5$.

CHAP. XII.

Of Square Roots, *and of* Irrational Numbers *resulting from them.*

123. What we have said in the preceding chapter amounts to this; that the square root of a given number is that number whose square is equal to the given number; and that we may put before those roots either the positive, or the negative sign.

124. So that when a square number is given, provided we retain in our memory a sufficient number of square numbers, it is easy to find its root. If 196, for example, be the given number, we know that its square root is 14.

Fractions, likewise, are easily managed in the same way. It is evident, for example, that $\frac{5}{7}$ is the square root of $\frac{25}{49}$; to be convinced of which, we have only to take the square root of the numerator and that of the denominator.

If the number proposed be a mixed number, as $12\frac{1}{4}$, we reduce it to a single fraction, which, in this case, will be $\frac{49}{4}$; and from this we immediately perceive that $\frac{7}{2}$, or $3\frac{1}{2}$, must be the square root of $12\frac{1}{4}$.

125. But when the given number is not a square, as 12, for example, it is not possible to extract its square root; or to find a number, which, multiplied by itself, will give the product 12. We know, however, that the square root of 12 must be greater than 3, because 3×3 produces only 9; and less than 4, because 4×4 produces 16, which is more than 12; we know also, that this root is less than $3\frac{1}{2}$, for we have seen that the square of $3\frac{1}{2}$, or $\frac{7}{2}$, is $12\frac{1}{4}$; and we may approach still nearer to this root, by comparing it with $3\frac{7}{15}$; for the square of $3\frac{7}{15}$, or of $\frac{52}{15}$, is $\frac{2704}{225}$, or $12\frac{4}{225}$; so that this

fraction is still greater than the root required, though but very little so, as the difference of the two squares is only $\frac{4}{225}$.

126. We may suppose that as $3\frac{1}{2}$ and $3\frac{7}{15}$ are numbers greater than the root of 12, it might be possible to add to 3 a fraction a little less than $\frac{7}{15}$, and precisely such, that the square of the sum would be equal to 12.

Let us therefore try with $3\frac{3}{7}$, since $\frac{3}{7}$ is a little less than $\frac{7}{15}$. Now $3\frac{3}{7}$ is equal to $\frac{24}{7}$, the square of which is $\frac{576}{49}$, and consequently less by $\frac{12}{49}$ than 12, which may be expressed by $\frac{588}{49}$. It is, therefore, proved that $3\frac{3}{7}$ is less, and that $3\frac{7}{15}$ is greater than the root required. Let us then try a number a little greater than $3\frac{3}{7}$, but yet less than $3\frac{7}{15}$; for example, $3\frac{5}{11}$; this number, which is equal to $\frac{38}{11}$, has for its square $\frac{1444}{121}$; and by reducing 12 to this denominator, we obtain $\frac{1452}{121}$ which shews that $3\frac{5}{11}$ is still less than the root of 12, viz. by $\frac{8}{121}$; let us therefore substitute for $\frac{5}{11}$ the fraction $\frac{6}{13}$, which is a little greater, and see what will be the result of the comparison of the square of $3\frac{6}{13}$, with the proposed number 12. Here the square of $3\frac{6}{13}$ is $\frac{2025}{169}$; and 12 reduced to the same denominator is $\frac{2028}{169}$; so that $3\frac{6}{13}$ is still too small, though only by $\frac{3}{169}$, whilst $3\frac{7}{15}$ has been found too great.

127. It is evident, therefore, that whatever fraction is joined to 3, the square of that sum must always contain a fraction, and can never be exactly equal to the integer 12. Thus, although we know that the square root of 12 is greater than $3\frac{6}{13}$, and less than $3\frac{7}{15}$, yet we are unable to assign an intermediate fraction between these two, which, at the same time, if added to 3, would express exactly the square root of 12; but notwithstanding this, we are not to assert that the square root of 12 is absolutely and in itself indeterminate: it only follows from what has been said, that this root, though it necessarily has a determinate magnitude, cannot be expressed by fractions.

128. There is therefore a sort of numbers, which cannot be assigned by fractions, but which are nevertheless determinate quantities; as, for instance, the square root of 12: and we call this new species of numbers, *irrational numbers*. They occur whenever we endeavour to find the square root of a number which is not a square; thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity. These numbers are also called *surd quantities*, or *incommensurables.*

129. These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes of which we may form an accurate idea; since, however concealed

the square root of 12, for example, may appear, we are not ignorant that it must be a number, which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, because it is in our power to approximate towards its value continually.

130. As we are therefore sufficiently acquainted with the nature of irrational numbers, under our present consideration, a particular sign has been agreed on to express the square roots of all numbers that are not perfect squares; which sign is written thus $\surd$, and is read *square root.* Thus, $\sqrt{12}$ represents the square root of 12, or the number which, multiplied by itself, produces 12; and $\sqrt{2}$ represents the square root of 2; $\sqrt{3}$ the square root of 3; $\sqrt{\frac{2}{3}}$ that of $\frac{2}{3}$; and, in general, $\sqrt{a}$ represents the square root of the number a. Whenever, therefore, we would express the square root of a number, which is not a square, we need only make use of the mark $\surd$ by placing it before the number.

131. The explanation which we have given of irrational numbers will readily enable us to apply to them the known methods of calculation. For knowing that the square root of 2, multiplied by itself, must produce 2; we know also, that the multiplication of $\sqrt{2}$ by $\sqrt{2}$ must necessarily produce 2; that, in the same manner, the multiplication of $\sqrt{3}$ by $\sqrt{3}$ must give 3; that $\sqrt{5}$ by $\sqrt{5}$ makes 5; that $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{2}{3}}$ makes $\frac{2}{3}$; and, in general, that $\sqrt{a}$ multiplied by $\sqrt{a}$ produces a.

132. But when it is required to multiply $\sqrt{a}$ by $\sqrt{b}$, the product is $\sqrt{ab}$; for we have already shewn, that if a square has two or more factors, its root must be composed of the roots of those factors; we therefore find the square root of the product ab, which is $\sqrt{ab}$, by multiplying the square root of a, or $\sqrt{a}$, by the square root of b, or $\sqrt{b}$; &c. It is evident from this, that if b were equal to a, we should have $\sqrt{aa}$ for the product of $\sqrt{a}$ by $\sqrt{b}$. But $\sqrt{aa}$ is evidently a, since aa is the square of a.

133. In division, if it were required, for example, to divide $\sqrt{a}$, by $\sqrt{b}$, we obtain $\sqrt{\frac{a}{b}}$; and, in this instance, the irrationality may vanish in the quotient. Thus, having to divide $\sqrt{18}$ by $\sqrt{8}$, the quotient is $\sqrt{\frac{18}{8}}$, which is reduced to $\sqrt{\frac{9}{4}}$, and consequently to $\frac{3}{2}$, because $\frac{9}{4}$ is the square of $\frac{3}{2}$.

134. When the number before which we have placed the radical sign $\surd$, is itself a square, its root is expressed in the

usual way; thus, $\sqrt{4}$ is the same as 2; $\sqrt{9}$ is the same as 3; $\sqrt{36}$ the same as 6; and $\sqrt{12\frac{1}{4}}$, the same as $\frac{7}{2}$, or $3\frac{1}{2}$. In these instances, the irrationality is only apparent, and vanishes of course.

135. It is easy also to multiply irrational numbers by ordinary numbers; thus, for example, 2 multiplied by $\sqrt{5}$ makes $2\sqrt{5}$; and 3 times $\sqrt{2}$ makes $3\sqrt{2}$. In the second example, however, as 3 is equal to $\sqrt{9}$, we may also express 3 times $\sqrt{2}$ by $\sqrt{9}$ multiplied by $\sqrt{2}$, or by $\sqrt{18}$; also $2\sqrt{a}$ is the same as $\sqrt{4a}$, and $3\sqrt{a}$ the same as $\sqrt{9a}$; and, in general, $b\sqrt{a}$ has the same value as the square root of bba, or $\sqrt{bba}$: whence we infer reciprocally, that when the number which is preceded by the radical sign contains a square, we may take the root of that square, and put it before the sign, as we should do in writing $b\sqrt{a}$ instead of $\sqrt{bba}$. After this, the following reductions will be easily understood:

$$\left.\begin{array}{l}\sqrt{8},\ \text{or}\ \sqrt{(2.4)}\\ \sqrt{12},\ \text{or}\ \sqrt{(3.4)}\\ \sqrt{18},\ \text{or}\ \sqrt{(2.9)}\\ \sqrt{24},\ \text{or}\ \sqrt{(6.4)}\\ \sqrt{32},\ \text{or}\ \sqrt{(2.16)}\\ \sqrt{75},\ \text{or}\ \sqrt{(3.25)}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}2\sqrt{2}\\ 2\sqrt{3}\\ 3\sqrt{2}\\ 2\sqrt{6}\\ 4\sqrt{2}\\ 5\sqrt{3}\end{array}\right.$$

and so on.

136. Division is founded on the same principles; as $\sqrt{a}$ divided by $\sqrt{b}$ gives $\frac{\sqrt{a}}{\sqrt{b}}$, or $\sqrt{\frac{a}{b}}$. In the same manner,

$$\left.\begin{array}{l}\frac{\sqrt{8}}{\sqrt{2}}\\ \frac{\sqrt{18}}{\sqrt{2}}\\ \frac{\sqrt{12}}{\sqrt{3}}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}\sqrt{\frac{8}{2}},\ \text{or}\ \sqrt{4},\ \text{or}\ 2\\ \sqrt{\frac{18}{2}},\ \text{or}\ \sqrt{9},\ \text{or}\ 3\\ \sqrt{\frac{12}{3}},\ \text{or}\ \sqrt{4},\ \text{or}\ 2.\end{array}\right.$$

Farther,
$$\left.\begin{array}{l}\frac{2}{\sqrt{2}}\\ \frac{3}{\sqrt{3}}\\ \frac{12}{\sqrt{6}}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}\frac{\sqrt{4}}{\sqrt{2}},\ \text{or}\ \sqrt{\frac{4}{2}},\ \text{or}\ \sqrt{2},\\ \frac{\sqrt{9}}{\sqrt{3}},\ \text{or}\ \sqrt{\frac{9}{3}},\ \text{or}\ \sqrt{3}.\\ \frac{\sqrt{144}}{\sqrt{6}},\ \text{or}\ \sqrt{\frac{144}{6}},\ \text{or}\ \sqrt{24},\end{array}\right.$$

or $\sqrt{(6 \times 4)}$, or lastly $2\sqrt{6}$.

137. There is nothing in particular to be observed in addition and subtraction, because we only connect the numbers by the signs + and −: for example, $\sqrt{2}$ added to $\sqrt{3}$ is

written $\sqrt{2} + \sqrt{3}$; and $\sqrt{3}$ subtracted from $\sqrt{5}$ is written $\sqrt{5} - \sqrt{3}$.

138. We may observe lastly, that in order to distinguish the irrational numbers, we call all other numbers, both integral and fractional, *rational numbers;* so that, whenever we speak of rational numbers, we understand integers, or fractions.

CHAP. XIII.

Of Impossible, *or* Imaginary Quantities, *which arise from the same source.*

139. We have already seen that the squares of numbers, negative as well as positive, are always positive, or affected by the sign $+$; having shewn that $-a$ multiplied by $-a$ gives $+aa$, the same as the product of $+a$ by $+a$: wherefore, in the preceding chapter, we supposed that all the numbers, of which it was required to extract the square roots, were positive.

140. When it is required, therefore, to extract the root of a negative number, a great difficulty arises; since there is no assignable number, the square of which would be a negative quantity. Suppose, for example, that we wished to extract the root of -4; we here require such a number as, when multiplied by itself, would produce -4: now, this number is neither $+2$ nor -2, because the square both of $+2$ and of -2, is $+4$, and not -4.

141. We must therefore conclude, that the square root of a negative number cannot be either a positive number or a negative number, since the squares of negative numbers also take the sign *plus:* consequently, the root in question must belong to an entirely distinct species of numbers; since it cannot be ranked either among positive or among negative numbers.

142. Now, we before remarked, that positive numbers are all greater than nothing, or 0, and that negative numbers are all less than nothing, or 0; so that whatever exceeds 0 is expressed by positive numbers, and whatever is less than 0 is expressed by negative numbers. The square roots of negative numbers, therefore, are neither greater nor less than nothing; yet we cannot say, that they are 0; for 0